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The algebras of meromorphic vector fields and their realisation on the spaces of meromorphic λ-differentials on Riemann surfaces I

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Abstract. The algebra of meromorphic vector fields with multipoles on a Riemann sphere is constructed explicitly. The central extension of this algebra is also given. The properties of meromorphic λ-differentials, as the realisation of the algebra, are investigated.

It is well known that the Virasoro algebra plays a very important role in string theory, conformal field theory and the other physical theories with conformal symmetry [1]. From global considerations, the Virasoro algebra without a central term is isomorphic to the algebra of meromorphic vector fields, which are holomorphic on the sphere S^2 except for the north pole and the south pole.

As a generalisation of the algebra of meromorphic vector fields with two poles on the sphere, in this paper we present the algebra of meromorphic vector fields with multipoles on the sphere. We will first construct this algebra and its central extension explicitly. Then we describe the modules over this algebra on the sphere, i.e. the λ-differentials with multipoles. Because of their great importance in physics, we pay special attention to the properties of the third-kind differentials on S^2.

In forthcoming papers, we will give the algebras of meromorphic vector fields with multipoles on compact Riemann surfaces with higher genus and the corresponding Kac–Moody algebra. We will also discuss the applications of the algebras in physics. It will be explained that the algebras are in fact generalisations of Krichever–Novikov algebras [2].

Let P_i $(i = 1, 2, \ldots, N)$ denote N different points on the Riemann sphere S^2. We fix the positions of the two points $z(P_1) = 0$ and $z(P_N) = \infty$. Let $\mathscr{H}_\lambda$ be the spaces of meromorphic λ-differentials ($\lambda \in Z$) holomorphic away from the points P_i on S^2. Let $D = \sum_{i=1}^{N} m_i P_i$ be an arbitrary divisor on the sphere. Let $\mathscr{H}_\lambda(D)$ be the spaces of meromorphic λ-differentials, whose value assignment at P_i is no less than $-m_i$. According to the Riemann–Roch theorem [3], we have

$$\dim \mathscr{H}_\lambda(D) = \dim \mathscr{H}_{1-\lambda}(-D) + (1 - 2\lambda) + \deg D. \tag{1}$$

In the case of $D = 0$, $\lambda = -1$, we find $\dim \mathscr{H}_2(0) = 0$, therefore the dimension of the holomorphic vector fields on S^2 is

$$\dim \mathscr{H}_{-1}(0) = 3. \tag{2}$$

Let L_0 and $L_{\pm 1}$ be a basis of $\mathscr{H}_{-1}(0)$. In the neighbourhood of the point P_1 these basis elements locally look like

$$L_{+1} = z^2\partial/\partial z \qquad L_{-1} = \partial/\partial z \qquad L_0 = z\partial/\partial z. \tag{3}$$

The algebra of these vector fields is just $\mathrm{sl}(2, c)$. The holomorphic basis also forms a subalgebra of $\mathscr{H}_{-1}$.

We set

$$\mathscr{H}^i_{-1}(m_i) \equiv \{H^i_{1-n},\ m_i \geq n \geq 2\} \tag{4}$$

where H^i_{1-n} are the meromorphic vector fields which have a pole of order $(n-1)$ at P_i $(1 \leq i < N)$, and a zero point of order $(n+1)$ at P_N. Thus the value assignments of H^i_{1-n} at points P_i and P_N can be respectively read as

$$\begin{aligned} V_{P_i}(H^i_{1-n}) &= 1 - n \qquad 1 \leq i < N \\ V_{P_N}(H^i_{1-n}) &= 1 + n. \end{aligned} \tag{5}$$

Near P_i, the vectors H^i_{1-n} can be locally expressed in the forms

$$H^i_{1-n} = (z - z_i)^{-n+1}\partial/\partial z \qquad 1 \leq i < N \tag{6}$$

and near P_N the vector H^N_{1-n} is given by

$$H^N_{1-n} = -\omega^{1-n}\partial/\partial\omega \qquad \omega = 1/z. \tag{7}$$

Its value assignments at P_i and P_N are given by

$$V_{P_N}(H^N_{1-n}) = 1 - n \qquad V_{P_i}(H^N_{1-n}) = 1 + n. \tag{8}$$

H^N_{1-n}, L_+, L_0, L_- and H^1_{1-n}, where $n \geq 2$, are also denoted collectively by H^1_{1-n} with $n \in Z$.

It is obvious that the meromorphic vectors L_0, $L_{\pm 1}$ and H^i_{1-n} $(i = 1, 2, \ldots, N;\ m_i \geq n \geq 2)$ are complex linear independent. They span a space of $\sum_{i=1}^N m_i + 3$ dimensions.

Based upon the formula (1), we obtain

$$\dim \mathscr{H}_{-1}(D) = \dim \mathscr{H}_2(-D) + 3 + \sum_{i=1}^{N} m_i \tag{9}$$

if $D = \sum_{i=1}^N m_i P_i$. Because of $\dim \mathscr{H}_2(-D) = 0$, we find

$$\dim \mathscr{H}_{-1}(D) = 3 + \sum_{i=1}^{N} m_i \tag{10}$$

Thus we conclude that $\{\bigcup_{i=1}^N \mathscr{H}^i_{-1}(m_i)\} \bigcup \mathscr{H}^0_{-1}$ is a basis of $\mathscr{H}_{-1}(D)$. Let m_i be large enough; it is easy to see that every meromorphic vector field, which only has poles at P_i $(i = 1,2,\ldots,N)$, can be expanded by the basis of $\mathscr{H}_{-1}(D)$. For $m_1, m_N \to \infty$, with respect to the basis $\mathscr{H}^0_{-1} \bigcup \mathscr{H}^1_{-1} \bigcup \mathscr{H}^N_{-1}$, the algebra is just the Virasoro algebra without central terms.

It can be shown that the algebra $\mathscr{H}_{-1}$ is independent of the positions of the poles. In other words the algebra of meromorphic vector fields holomorphic outside P_i, $(i = 1,2,\ldots,N)$ is isomorphic to the algebra of meromorphic vector fields holomorphic outside Q_i, $(i = 1,2,\ldots,N)$. Therefore we may take an appropriate choice of the positions of poles such that $z(P_i) = i-1$, $i = 1,2,\ldots,N-1$ and $z(P_N) = \infty$. Then the algebraic commutation relations can be given explicitly by:

$$[H^i_{1-n}\,,\ H^i_{1-m}] = (n-m)H^i_{1-(n+m)} \tag{11}$$

where $i = 1,2,\ldots,N-1$, $n,m \geq 2$, and when $i = 1$, $n,m \in Z$;

$$[H^i_{1-n}, H^j_{1-m}] = \sum_{\alpha=2}^{n+1} A^{ij}_{nm}(\alpha)H^i_{1-\alpha} - \sum_{\beta=2}^{m+1} B^{ij}_{nm}(\beta)H^j_{1-\beta} \tag{12}$$

where $i < j$, $n,m \geq 2$, $i,j = 1,2,\ldots,N-1$, and

$$A^{ij}_{nm}(\alpha) = (2n-\alpha)C^{n+1-\alpha}_{1-m}(i-j)^{\alpha-m-n}$$
$$B^{ij}_{nm}(\beta) = (2m-\beta)C^{m+1-\beta}_{1-n}(j-i)^{\alpha-m-n};$$

$$[H^1_{1-n},\ H^j_{1-m}] = \sum_{\alpha\leq 1}^{n+1} C^{1j}_{nm}(\alpha)H^1_{1-\alpha} - \sum_{\beta=2}^{m+1} D^{1j}_{nm}(\beta)H^j_{1-\beta} \tag{13}$$

where $n \leq 1$, $m \geq 2$, $j = 2,3,\ldots,N-1$, and

$$C^{1j}_{nm}(\alpha) = (2n-\alpha)C^{\alpha-m-n}_{1-m}(-j)^{\alpha-m-n} \qquad D^{1j}_{nm}(\beta) = B^{1j}_{nm}(\beta).$$

Let $\tilde{\mathscr{H}}_{-1}$ be the central extension algebra of $\mathscr{H}_{-1}$; the commutation relations for $\tilde{\mathscr{H}}_{-1}$ are now:

$$[\tilde{H}^i_{1-n},\ \tilde{H}^i_{1-m}] = (n-m)\tilde{H}^i_{1-(n+m)} - c\delta_{i,0}\delta_{m+n,0}(n^3-n) \tag{14}$$

where $m,n \geq 2$, $i = 1,2,\ldots,N-1$, and when $i = 1$, $n,m \in Z$;

$$[\tilde{H}^i_{1-n},\ \tilde{H}^j_{1-m}] = \sum_{\alpha=2}^{n+1} A^{ij}_{nm}(\alpha)\tilde{H}^i_{1-\alpha} - \sum_{\beta=2}^{m+1} B^{ij}_{nm}(\beta)\tilde{H}^j_{1-\beta} - c\delta_{i,1}C^{1+n}_{1-m}(-z_j)^{\alpha-m-n}(n^3-n) \tag{15}$$

where $n,m \geq 2$, $i < j$ and $i,j = 1,2,\ldots,N-1$;

$$[\tilde{H}^1_{1-n},\ \tilde{H}^j_{1-m}] = \sum_{\alpha\leq 1}^{n+1} C^{1j}_{nm}(\alpha)\tilde{H}^1_{1-n} - \sum_{\beta=2}^{m+1} D^{1j}_{nm}(\beta)\tilde{H}^j_{1-\beta} \tag{16}$$

where $n \leq 1$, $m \geq 2$, $j = 2,3,\ldots,N-1$, and

$$[c,\ \tilde{H}^i_{1-n}] = 0 \qquad i = 1,2,3,\ldots,N-1.$$

(Note that

$$C^M_K = \frac{K(K-1)(K-2)\ldots(K-M+1)}{K!} \tag{17}$$

and when $M < 0$ or $M > K > 0$, $C_K^M = 0$.)

By means of the method mentioned above, we can also investigate the meromorphic λ-differentials and construct bases of $\mathscr{H}_\lambda$. Let

$$\begin{aligned} &\mathscr{H}_\lambda^i \equiv \{\phi_{i,n}^\lambda;\ n \le -1\} \qquad i = 1,2,\ldots,N-1 \\ &V_{P_i}(\phi_{i,n}^\lambda) = n \qquad n \le -1 \\ &V_{P_N}(\phi_{i,n}^\lambda) = -n - 2\lambda \end{aligned} \tag{18}$$

where $\phi_{i,n}^\lambda$ are the meromorphic λ-differentials holomorphic outside P_i and P_N. Similarly we have

$$\mathscr{H}_\lambda^N \equiv \{\phi_{N,n}^\lambda;\ n \le -2\lambda\} \qquad V_{P_i}(\phi_{N,n}^\lambda) = -n - 2\lambda \qquad V_{P_N}(\phi_{N,n}^\lambda) = n. \tag{19}$$

Observe that these meromorphic λ-differentials are complex linear independent. Let $D = \sum_{i=1}^N m_i P_i$ be a divisor. In terms of the Riemann–Roch theorem, we know the meromorphic λ-differentials

$$\begin{aligned} &\phi_{i,n}^\lambda \qquad -m_i \le n \le -1 \qquad i = 1,2,\ldots,N-1 \\ &\phi_{N,n}^\lambda \qquad -m_N \le n \le -2\lambda \end{aligned} \tag{20}$$

form a complex linear space of $(\sum_{i=1}^N m_i + 1 - 2\lambda)$ dimensions. Generally speaking we can introduce a set of meromorphic λ-differentials

$$\begin{aligned} &\phi_{i,n}^\lambda \qquad n \le -1 \qquad i = 1,2,\ldots,N-1 \\ &\phi_{N,n}^\lambda \qquad n \le -2\lambda \end{aligned} \tag{21}$$

which is a basis of the infinite-dimensional space $\mathscr{H}_\lambda$ for a fixed λ. Near P_i or P_N the elements of the basis can be locally expressed in the forms

$$\begin{aligned} &\phi_{i,n}^\lambda = (z - z_i)^n (\mathrm{d}z)^\lambda \qquad n \le -1 \qquad i = 1,2,\ldots,N-1 \\ &\phi_{N,n}^\lambda = w^n (\mathrm{d}w)^\lambda \qquad n \le -2\lambda \end{aligned} \tag{22}$$

where $w = 1/z$ and $z(P_1) = z_1 = 0$, $z(P_i) = z_i$, $z(P_N) = \infty$.

The algebra $\mathscr{H}_{-1}$ acts naturally on the spaces $\mathscr{H}_\lambda$. For example $(i,\ j > 1)$,

$$\begin{aligned} H_{1-n}^i \phi_{j,-m}^\lambda = {} & (\alpha + \lambda(1-n))(-1)^m \sum_{\alpha=0}^{n-1} C_{m+\alpha-1}^\alpha (j-i)^{-m-\alpha} \phi_{i,\alpha-n}^\lambda \\ & + (\lambda\alpha - m)(-1)^{n+1} \sum_{\beta=0}^{m} C_{n+\beta-2}^\beta (i-j)^{-n+1-\beta} \phi_{j,\beta-m-1}^\lambda . \end{aligned} \tag{23}$$

Hence $\mathscr{H}_\lambda$ are generalised–graded modules over $\mathscr{H}_{-1}$.

An inner product of meromorphic λ-differentials may be defined by

$$\langle \varphi_i^\lambda,\ \varphi_j^\lambda \rangle = \lim_{r_i \to 0} \int_{S^2 | \bigcup_{i=1}^N D_i} \int \varphi^i_{\underbrace{z z \ldots z}_{\lambda}} \bar{\varphi}^j_{\underbrace{\bar{z}\bar{z} \ldots \bar{z}}_{\lambda}} (K^z k^{\bar{z}})^\lambda \omega \wedge \bar{\omega} \tag{24}$$

where

$$\varphi_i^\lambda = \varphi^i_{\underbrace{z z \ldots z}_{\lambda}} (\mathrm{d}z)^\lambda \in \mathscr{H}_\lambda$$

and $\omega = K_z\,\mathrm{d}z$ is an Abelian differential of the third kind holomorphic outside P_i which can be uniquely fixed if the residues of ω have been determined, D_i is a small disc whose centre is at P_i with radius r_i.

In finite dimensions, applying the inner product and the standard Gram–Schmidt orthogonalising process we may obtain a new basis of $\mathscr{H}_\lambda$

$$\begin{array}{lll} \varphi_{i,n}^\lambda & n \leq -1 & i = 1,2,\ldots,N-1 \\ \varphi_{N,n}^\lambda & n \leq -2\lambda & \end{array} \tag{25}$$

which are mutually orthogonal to each other and

$$\langle \varphi_{i,m}^\lambda,\ \varphi_{j,m}^\lambda \rangle = \delta_{i,j}\delta_{m,n}. \tag{26}$$

An arbitrary element $X^\lambda \in \mathscr{H}_\lambda$ can be expanded in $\{\varphi_{i,m}^\lambda\}$:

$$X^\lambda = \sum_{m,i} A_{i,m}\varphi_{i,m}^\lambda \tag{27}$$

$$A_{i,m} = \langle X^\lambda,\ \varphi_{i,m}^\lambda \rangle. \tag{28}$$

In fact the calculation of the formulae (27) and (28) is closely related to the properties of the third-kind Abelian differentials ω. In addition to this we may also introduce the concept of Euclidean time on the sphere by use of ω, which has simple poles at P_i, $i = 1,2,\ldots,N$, and is holomorphic away from P_i. Define $\tau(Q)$

$$\tau(Q) = \mathrm{Re}\int_{Q_0}^{Q} \omega \tag{29}$$

where Q_0 is an arbitrary initial point on S^2 and the point Q is movable. Because ω has periods purely imaginary along any cycle on S^2, the function τ is a univalent function independent of the path connecting Q_0 with Q. Thus it is reasonable to generalise the concept of level lines, i.e. $C_\tau = \{Q \in S^2 \mid \tau(Q) = t\}$, which is suggested by Krichever and Novikov in the case of two distinguishable points $P_\pm$ on a Riemann surface.

Without loss of generality, we give here an example of ω, which has the simple poles at three points P_1, P_2 and P_N. Such a third-kind differential can be locally expressed as

$$\omega = \frac{z-2}{z(z-1)}\,\mathrm{d}z. \tag{30}$$

One can see that C_τ becomes one small circle around P_1 as τ goes to $-\infty$, and two small circles around P_2 and P_N respectively as τ goes to $+\infty$. The remarkable fact is that C_τ is not always connected, but can split into two components in our case. To see this, let us perform the integration of (29)

$$\begin{aligned} \tau &\equiv \mathrm{Re}\int_{Q_0}^{Q} \omega = \frac{1}{2}\int_{Q_0}^{Q} (\omega + \bar{\omega}) \\ &= \frac{1}{2}\left[\ln\left(\frac{(z(Q_0)-1)(\bar{z}(Q_0)-1)}{z^2(Q_0)\bar{z}^2(Q_0)}\right) + \ln\left(\frac{z^2(Q)\bar{z}^2(Q)}{(z(Q)-1)(\bar{z}(Q)-1)}\right)\right]. \end{aligned} \tag{31}$$

Choosing the initial point Q_0 such that the first term is equal to zero, we find

$$P_i = \begin{cases} 0 & \tau \to -\infty \\ (1, \infty) & \tau \to \infty. \end{cases}$$

Because $\tau = \ln |z^2(P)/(z(P)-1)|$, $z(P)$ is determined up to a phase factor if τ goes to $\pm\infty$. Therefore the level lines C_τ are some small circles with centres 0, 1, ∞ respectively as $\tau \to \pm\infty$.

Set

$$\frac{z^2(P)}{z(P)-1} = A. \tag{32}$$

In terms of polar coordinates we can rewrite (32) in the following forms:

$$\begin{aligned} r^2 \cos 2\theta_1 &= ar\cos(\theta_1+\theta_2) - a\cos\theta_2 \\ r^2 \sin 2\theta_1 &= ar\sin(\theta_1+\theta_2) - a\sin\theta_2 \end{aligned} \tag{33}$$

where $z = r\,e^{i\theta_1}$ and $A = a\,e^{i\theta_2}$. It is easy to see that the number of real solutions r depends on θ_1, i.e. $r = r(\theta_1)$. In other words, the number of cross points of meridian with a fixed θ_1 passing through the level lines is just the number of solutions r of (33). If the meridian passes through the split point of two closed level lines, the meridian must be a tangent to both circles at the same point. This means that there is a twofold root for (31) at the split point. We find that $r_1 = r_2 = 2$ if $\theta_1 = 0$, $a = 4$. This is just the zero point $z = 2$ of ω. Since $\partial\tau/\partial z(Q)$ has a minimal value at the point $z(Q) = 2$, it is understood that $\tau(Q)$ is a Morse function. This conclusion is also true for the general case.

The picture of level lines on the sphere described by the third-kind Abelian differential with three simple poles is especially interesting in string theory. The basic interaction diagram of the closed string is that one string is produced in the infinite past, then splits into two strings at a point in spacetime, and finally the two new strings vanish in the infinite future. In some forthcoming papers we will investigate the role of the algebra $\tilde{\mathscr{H}}_{-1}$ of meromorphic vector fields holomorphic outside P_i ($i = 1, 2, \ldots, N$) on S^2 in physics, and discuss the relations between the string interaction vertex and Abelian differentials of the third kind.

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